Preliminaries 1 for BCHM

Sridhar Venkatesh

Outline

Nakayama-Zariski decomposition

Definition-Lemma 3.3.1

$X =$ sm. proj, $B =$ big $\mathbb{R}$-divisor, $C =$ prime divisor.

$$\sigma_C(B) = \inf\{\mathrm{mult}_C(B') \mid B \sim_{\mathbb{R}} B' \geq 0\}$$

Then, $\sigma_C =$ cont. function on cone of big divisors. In fact, σ_C extends to the boundary as follows:

$$\sigma_C(D) = \lim_{\epsilon \to 0} \sigma_C(D + \epsilon A) \text{ for } A \text{ ample}$$

For a given D, there are only finitely many C s.t. $\sigma_C(D) > 0$.Set:

$$N_\sigma(D) = \sum_C \sigma_C(D) C$$

$$\implies D = N_\sigma(D) + (D - N_\sigma(D))$$

$$\implies D = \text{ 'Negative'} + \text{'Positive'}$$

Proposition 3.3.2

'The positive part has sections'

$X =$ sm. proj, $D =$ pseudo-eff $\mathbb{R}$-divisor, $B =$ any big $\mathbb{R}$-divisor.
If $P := D - N_\sigma(D) \not\equiv 0$, then $\exists$ positive k, β s.t.:

$$h^0(\mathcal{O}_X(\lfloor mP \rfloor + \lfloor kB \rfloor)) > \beta m \text{ for all } m \gg 0$$

In particular:

$$h^0(\mathcal{O}_X(\lfloor mD \rfloor + \lfloor kB \rfloor)) > \beta m \text{ for all } m \gg 0$$

Basic Facts about Adjunction

Definition-Lemma 3.4.1

(X, Δ) log canonical.
S = normal comp of $\lfloor \Delta \rfloor$ with coeff $= 1$.
Θ = Divisor on S defined by $(K_X + S)|_S = K_S + \Theta$.

1. (X, Δ) dlt $\implies$ $(K_S + \Theta)$ dlt.
2. (X, Δ) plt $\implies$ $(K_S + \Theta)$ klt.
3. $(X, \Delta = S)$ plt $\implies$ coeff of any D in Θ is of the form $\frac{r-1}{r}$ where r = index of S at μ_D.
4. (X, Δ) plt $\implies$ 'Adjunction behaves well under projective birational maps'.
   Let $f : Y \to X$ projective birational, let Δ_Y, Θ_Y defined by:

$$K_Y + \Delta_Y = f^*(K_X + \Delta), (K_Y + \Delta_Y)|_{\tilde{S}} = K_{\tilde{S}} + \Theta_Y$$

   Then we have:

$$(f|_{\tilde{S}})_*(\Theta_Y) = \Theta$$

Stable Base Locus

Notions for $\mathbb{R}$-divisors

$\pi : X \to U$ projective morphism of normal varieties, $D = \mathbb{R}$-divisor on X.

Definition

1. The **real linear system** associated to D over U is:

$$|D/U|_{\mathbb{R}} := \{C \text{ effective} \mid C \sim_{\mathbb{R},\pi} D\}$$

2. The **stable base locus** is:

$$B(D/U) := \bigcap_{C \in |D/U|} Supp(C)$$

3. The **stable fixed divisor** is the divisorial support of $B(D/U)$.
4. The **augmented base locus** is:

$$B_+(D/U) := B((D - \epsilon A)/U) \text{ for } \epsilon \ll 1, A \text{ ample}$$

Remark

1. Agrees with the usual definition when D is a $\mathbb{Z}$-divisor. (**Idea:** Given $x \in X$, need to prove:

$$\exists\ \mathbb{R}\text{-divisor } D_{\mathbb{R}} \in B(D/U)_{\mathbb{R}} \text{ not passing thru } x \implies \exists\ \mathbb{Q}\text{-divisor } D_{\mathbb{Q}} \in B(D/U)_{\mathbb{Q}} \text{ not passing thru } x$$

   We do the following:
   - Look at a suitable subcone $W \subset \mathrm{WDiv}_{\mathbb{R}}(X)$ of all $D' \in |D/U|_{\mathbb{R}}$ not passing thru x.
   - W will be generated by finitely many $\mathbb{Z}$-divisors, so W is a rational polyhedron.
   - W is non-empty since we have $D_{\mathbb{R}} \in W$. Thus W has a $\mathbb{Q}$-point i.e. $\exists$ a $\mathbb{Q}$-divisor $D_{\mathbb{Q}} \in B(D/U)_{\mathbb{Q}}$ not passing thru x.
2. Like in the $\mathbb{Q}$-divisor case, these are only defined as closed subsets.

Useful Lemma

We're working towards decomposing every divisor as 'Movable + Fixed'.

Lemma 3.5.6

Let $D \geq 0$ be an $\mathbb{R}$-divisor.
Assume $\exists\ D' \in |D/U|_{\mathbb{R}}$ which has no common components with D.
Then we can find $D'' \in |D/U|_{\mathbb{R}}$ s.t.:

A multiple of every component of D'' is mobile.

This is saying: If you can move D to avoid the components of D, then you can move D to make every component mobile.

Every Divisor = Movable + Fixed

Proposition 3.5.4

Say $D \geq 0$. Then $\exists$ $\mathbb{R}$-divisors $M, F \geq 0$ s.t.:

1. $D \sim_{\mathbb{R},\pi} M + F$.
2. $\mathrm{Supp}(F) \subset B(D/U)$.
3. If B is a component of M, then some multiple of B is mobile.

Thus, '$D =$ Movable + Fixed'.

Proof

Write $D = M + F$ where:

- F is contained in $B(D/U)$.
- No component of M is contained in $B(D/U)$.

Call a prime divisor **bad** if no multiple is mobile.

Proof of Proposition

Proof cont.

We prove by induction on the number of bad components of M.

- Let B be a bad component of M. We will find $D' \in |D/U|$ s.t.
  - Bad components of $M' \subset$ Bad components of M.
  - B is no longer a component of D'.
- $B \not\subset B(D/U)$ and so, $\exists\ D_1 \in |D/U|_{\mathbb{R}}$ s.t. $B \not\subset D_1$.
- Take $E = D \wedge D_1$ (common components of D and D_1). Then $D - E \sim_{\mathbb{R}} D_1 - E$ are effective and have no common components.
- Lemma $\implies$ Get a $D'' \in |(D-E)/U|$ which does not have bad components.
- $\therefore$ Only bad components of $D'' + E \in |D/U|$ are among E, hence among D. Also, $B \not\subset E$. Done!

$\square$

Types of Models

Negativity Lemma

Lemma 3.6.2

$f : Y \to X$ be a proj birational map of normal quasi-proj varieties.
$D = \mathbb{R}$-Cartier divisor on Y s.t. $-D$ is f-nef. Write:

$$D = D_{\text{horizontal}} + D_{f\text{-exceptional}}$$

Then:

$$D_{\text{horizontal}} \geq 0 \implies D_{f\text{-exceptional}} \geq 0$$

We keep cutting by hyperplanes in X and reduce to $X =$ surface. There, it follows from the Hodge Index Theorem.

Example

$f : \mathrm{Bl}_0 \mathbb{P}^2 \to \mathbb{P}^2$. Take $D = E$.
$E^2 = -1$ and $C.E \geq 0$ for every other divisor C.

D-non-positive and D-negative

Definition

$\phi : X \dashrightarrow Y$ proper birational contraction of normal quasi proj. var.
$D = \mathbb{R}$-Cartier divisor on X s.t. $D' = \phi_* D$ is also $\mathbb{R}$-Cartier.

- We say ϕ is D-**non-positive** if for some common resolution $p : W \to X$, $q : W \to Y$, we have:

$$p^* D = q^* D' + E$$

  where E is effective, q-exceptional.
- We say ϕ is D-**negative** if additionally $\mathrm{Supp}(E)$ contains the strict transform of the ϕ-exceptional divisors.

By Negativity Lemma, can replace 'E effective, q-exceptional' with '$p_* E$ effective'.

Models

$\pi : X \to U$ proj. morphism of normal varieties, $D = \mathbb{R}$-Cartier on X.
Say that a birational contraction $f : X \dashrightarrow Y$ over U is a **semi-ample model** of D over U if:

- Y is normal and projective over U.
- f is D-non-positive.
- $f_* D$ is semiample over U

Say that a rational map $g : X \dashrightarrow Z$ over U is the **ample model** of D over U if:

- Z is normal and projective over U.
- If $p : W \to X$ and $q : W \to Z$ resolve g, then q is a contraction.
- $\exists$ ample divisor H over U on Z s.t. we may write $p^* D \sim_{\mathbb{R},\pi} q^* H + E$ where $E \geq 0$ and E lies in the stable base locus of $p^* D$ over U.

Facts about semi-ample and ample models

1. **'Ample models are unique':** If $g_i : X \dashrightarrow X_i$ are two ample models, then $\exists$ an isomorphism $\chi : X_1 \to X_2$ s.t. $g_2 = \chi \circ g_1$.
   - Let $g : Y \to X$ resolve the indeterminacies of g_i and let $f_i = g_i \circ g$ be the induced contractions.
   - Have: $g^*D = f_i^*H_i + E_i$ and E_i lies in the stable base locus of g^*D.
   - $E_1 \subset B(g^*D/U) = B((f_2^*H_2 + E_2)/U) \subset E_2$ (as H is ample).
   - Thus $E_1 \leq E_2$. By symmetry, $E_1 = E_2$.
   - Thus $f_1^*H_1 \sim_{\mathbb{R},\pi} f_2^*H_2$. Thus, $f_1 = f_2$ as they contract the same curves.
2. Suppose $g : X \dashrightarrow Z$ is an ample model, then we can write $p^*D \sim_{\mathbb{R},\pi} q^*H + E$ where $E \geq 0$ and if F is any p-exceptional divisor whose centre lies in the indeterminacy locus of g then F is contained in $\mathrm{Supp}(E)$.
   - This is an application of Negativity Lemma.

3. **'Semiample model exists $\implies$ Ample model exists':** If $f : X \dashrightarrow Y$ is a semiample model of D over U, then $\exists$ a contraction $h : Y \to Z$ s.t. $h \circ f : X \dashrightarrow Z$ is an ample model. Additionally, $f_* D \sim_{\mathbb{R},\pi} h^* H$.
   - Remember $f_* D$ is semiample over U.
   - Let $h : Y \to Z$ be the morphism over U defined by $f_* D$. We can check that this gives us the ample model for X over U.
4. **'In the birational case, ample model is exactly analogous to semiample model':** If $f : X \dashrightarrow Y$ is a birational contraction over U, then f is the ample model $\iff$ f is a semiample model and $f_* D$ is ample over U.
   - ($\impliedby$) By (3), we know we can contract $h : Y \to Z$ to get an ample model Z. Additionally, $f_* D \sim_{\mathbb{R},\pi} h^* H$.
   - But $f_* D$ is ample over U and so $h^* H$ is ample over U.
   - Pullback under contraction h is ample $\implies$ h doesn't contract any curves i.e. h is an isomorphism.

More models

$\pi : X \to U$, $Y \to U$ be proj. morphisms of normal, quasi-proj. varieties.
Let $\phi : X \dashrightarrow Y$ be a birational contraction.
Assume $K_X + \Delta$ log canonical. Set $\Gamma = \phi_* \Delta$.

1. Y is a **log terminal model** for $K_X + \Delta$ over U if ϕ is $(K_X + \Delta)$-negative, $K_Y + \Gamma$ is dlt and nef over U, and Y is $\mathbb{Q}$-factorial.
   (Modern name = Minimal Model)
2. Y is a **weak log canonical model** for $K_X + \Delta$ over U if ϕ is $(K_X + \Delta)$-non-positive, and $K_Y + \Gamma$ is nef over U.
   (Modern = Minimal Model + Flops)
3. Y is the **log canonical model** for $K_X + \Delta$ over U if ϕ is the ample model of $K_X + \Delta$ over U.
   (Modern name = Ample Model)
4. Y is a **good minimal model** if $K_Y + \Gamma$ is semiample.

Diagram of different models

$$X \overset{K_X\text{-neg}}{\dashrightarrow} X_1 \dashrightarrow \cdots \dashrightarrow X_{min}$$

($K_{X_{min}}$ nef)

(log terminal model)

$$X_{min} \xrightarrow{\text{(Assuming } R(X, K_X) \text{ is fin. gen.)}} X_{amp}$$

X_{amp} (ample model / canonical model)

$\left(+ K_{X_{min}} \text{ big} \overset{\text{Abundance}}{\Longrightarrow} \text{Semiample model}\right)$

$$\begin{tikzcd} X \arrow[r, dashed, "K_X\text{-neg}"] \arrow[rr, dashed, bend right, "K_X\text{- non-positive}"'] & X_{min} \arrow[r, dashed, "\text{Flop}"] & X'_{min} \end{tikzcd}$$

[Weak lc model

Flop must happen away from MMP loci]

More lemmas about these models

Lemma 3.6.8

'Weak lc models and lt models are preserved under taking positive multiples of $K_X + \Delta$.'

$\phi : X \dashrightarrow Y$ be a birational contraction over U.
(X, Δ) and (X, Δ') two log pairs. Set $\Gamma := f_*\Delta$ and $\Gamma' := f_*\Delta'$.
$\mu > 0$ positive real number.

- $K_X + \Delta$, $K_X + \Delta'$ lc. $(K_X + \Delta') \sim_{\mathbb{R},\pi} \mu(K_X + \Delta)$.

  $$\phi \text{ weak lc model for } K_X + \Delta \iff \phi \text{ weak lc model for } K_X + \Delta'$$

- $K_X + \Delta$, $K_X + \Delta'$ klt. $(K_X + \Delta') \equiv_\pi \mu(K_X + \Delta)$.

  $$\phi \text{ lt model for } K_X + \Delta \iff \phi \text{ lt model for } K_X + \Delta'$$

For example, both conditions say $K_Y + \Gamma$ nef $\iff$ $K_Y + \Gamma'$ nef.

Lemma 3.6.9

'Composition of lt models is a lt model.'

$\phi : X \dashrightarrow Y$ lt model of (X, Δ), $\varphi : Y \dashrightarrow Z$ lt model of $(Y, \phi_* \Delta)$.
Then:

$$\eta := \varphi \circ \phi \text{ lt model of } (X, \Delta)$$

Proof

- Clear that η is a birational contraction, Z is $\mathbb{Q}$-factorial and $K_Z + \eta_* Z$ is dlt and nef over U.
- Only thing to show is that η is $K_X + \Delta$-negative.

(cont. in next page)

Proof cont.

Take a common resolution:

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "r"] & \\ X \arrow[r, dashed, "\phi"] & Y \arrow[r, dashed, "\varphi"] & Z \end{tikzcd}$$

ϕ lt model $\implies \phi$ is $K_X + \Delta$-negative $\implies$
$p^*(K_X + \Delta) - q^*(K_Y + \phi_*\Delta) = E_1 \geq 0$, and $\mathrm{Supp}(E_1) = \mathrm{Exc}(\phi)$.
φ lt model $\implies \varphi$ is $K_Y + \phi_*\Delta$-negative $\implies$
$q^*(K_Y + \phi_*\Delta) - r^*(K_Z + \eta_*\Delta) = E_2 \geq 0$, and $\mathrm{Supp}(E_2) = \mathrm{Exc}(\varphi)$.

$$\begin{aligned} p^*(K_X + \Delta) - r^*(K_Z + \eta_*\Delta) &= p^*(K_X + \Delta) - q^*(K_Y + \phi_*\Delta) \\ &\quad + q^*(K_Y + \phi_*\Delta) - r^*(K_Z + \eta_*\Delta) \\ &= E_1 + E_2 \geq 0 \end{aligned}$$

And $\mathrm{Supp}(E_1 + E_2) = \mathrm{Exc}(\eta)$. Thus η is $K_X + \Delta$-negative. □

Lemma 3.6.10

'Suitable lt model of a resolution of X is also a lt model of X'

(X, Δ) klt with Δ big over U.
$f : Z \to X$ any log resolution of (X, Δ). Write:

$$K_Z + \Phi_0 = f^*(K_X + \Delta) + E$$

where E, Φ_0 effective and have no common components, $f_*\Phi_0 = \Delta$ and E is exceptional.
Let $F \geq 0$ be any divisor with $\mathrm{Supp}(F) = \mathrm{Exc}(f)$.
If $\eta > 0$ is sufficiently small and $\Phi = \Phi_0 + \eta F$, then $K_Z + \Phi$ is klt and Φ is big over U. Moreover:

$$Z \dashrightarrow W \text{ lt model of } K_Z + \Phi \implies X \dashrightarrow W \text{ lt model for } K_X + \Delta.$$

Lemma 3.6.11

Fix $\phi : X \dashrightarrow Y$. Then:

$$\{\Delta \mid \phi \text{ is a weak lc model for } (X, \Delta)\} = \overline{\{\Delta \mid \phi \text{ is an ample model for } (X, \Delta)\}}$$

$X = \mathbb{Q}$-factorial. (X, Δ) dlt. Write $\Delta = S + B$ where $S := \lfloor \Delta \rfloor$.
$\phi : X \dashrightarrow Y$ weak lc model of (X, Δ).
Suppose that the components of B span $(\mathrm{WDiv}_{\mathbb{R}}(X)/\equiv)$.
Let V be any finite dimensional affine subspace of $\mathrm{WDiv}_{\mathbb{R}}(X)$ which contains the subspace generated by the components of B.
Then:

$$\mathcal{W}_{\phi,S,\pi}(V) = \overline{\mathcal{A}_{\phi,S,\pi}(V)}$$

$$\mathcal{W}_{\phi,S,\pi}(V) := \{\Delta' = S + B' \text{ for } B' \in V, B' \geq 0 \mid K_X + \Delta' \text{ is lc, pseudo-eff,}$$
$$\phi \text{ is a weak lc model for } (X, \Delta')\}$$

$\mathcal{A}_{\phi,S,\pi}(V)$ is defined similarly for ample models.